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Elastic behavior in Contact Dynamics of rigid particles

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The systematic errors due to the practical implementation of the Contact Dynamics method for simulation of dense granular media are examined. It is shown that, using the usual iterative solver to simulate a chain of rigid particles, effective elasticity and sound propagation with a finite velocity occur. The characteristics of these phenomena are investigated analytically and numerically in order to assess the limits of applicability of this simulation method and to compare it with soft particle molecular dynamics.

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I. INTRODUCTION

In computational physics one can distinguish two different validation tasks, which have to be solved in order to make simulations a useful research tool. First one must prove the validity of a simulation model by comparing its results to laboratory experiments, and second, equally important, one must assess the systematic errors due to the practical implementation in order to tell, how precisely the simulation results reflect the theoretical properties of the simulation model. In this paper we address the second type of validation problem for the simulation technique of *contact dynamics*, which was developed about 10 years ago [1–3] with the aim to investigate granular media [4] in the limit of high rigidity of the particles at a high packing density. This method has been successfully applied and reproduces experiments (see e.g. [5]). However, its systematic errors and the computational effort to keep them tolerably small have not been investigated in detail before.

This is in marked contrast to other discrete elements methods for granular media (cf. e.g. [6]), in particular the soft particle *molecular dynamics* simulation model, which has been widely used since more than 20 years [7]. During this time possible pitfalls such as the *detachment effect*[8] and the *brake failure effect*[9] could be discovered, analyzed and hence avoided.

Contact dynamics simulations have been applied to study a large variety of questions in dense granular systems, where excluded volume interactions and static friction, so called unilateral constraints, are believed to be essential [10–14], but it should be mentioned that such constraints arise also in other areas like virtual reality, engineering, especially in robotics, and operations research, where the numerical treatments are similar [15, 16].

In a system of perfectly rigid particles the sound velocity would be infinite. This is in principle borne out by the contact dynamics simulation model. However, its practical implementation will normally give rise to sound-waves in the granular material, as we are going to show in the following, even if each single collision is modeled as being perfectly inelastic. Our aim is to elucidate this artifact after introducing briefly the principles of this method.

II. THE CONTACT DYNAMICS METHOD

First, let us point out the basic difference between molecular dynamics (MD) on one side and contact dynamics (CD) on the other. Both have in common the integration of Newton's equation of motion where the occurring forces are due to external fields (gravity) or are – more important – *contact forces*, i.e. caused by contacts between particles or their contacts with confining walls.

The spirit of MD is to calculate the contact forces according to their *cause*, i.e. the (usually microscopic) deformation of the contact region and the involved velocities. Since the full treatment of every particle as a deformable body would render the simulation of a large number of particles exceedingly time consuming, a lot of models exist, how to replace the deformation by the local overlap [17], the latter being a virtual quantity obtained from the undeformed shapes.

The principles of CD are different: Here the contact forces are calculated by virtue of their *effect*, which is to fulfill certain constraints. Typically, such a constraint is the volume exclusion of the particles or the absence of sliding due to static friction. As can be seen immediately, this problem cannot be solved locally: In a cluster of particles where many contacts are simultaneously present, the force at one contact depends on adjacent contacts and so on. In that case the aim is to find a global force-network, which is consistent with the constraints at all contacts. The method to carry out this calculation is often called the *solver* in this context, which is commonly an iterative scheme, as the one we describe in the following section. In order to make our points very clear, we perform an analytical investigation for a one dimensional example but can prove the existence of the discovered effect in two dimensions as well.

A. The dynamical equations

As CD is designed to obey excluded volume constraints exactly, particle collisions lead to discontinuous velocity changes ("shocks"), i.e. to nonsmooth mechanics [18]. Therefore, higher order terms than employed in the Euler integration scheme are of no use. For the ith particle's positions x_i, this reads

$$x_i(t+\Delta t) = x_i(t) + v_i(t+\Delta t)\Delta t \ , \tag{1}$$

and correspondingly for its velocity v_i we have

$$v_i(t+\Delta t) = v_i(t) + \frac{F_i(t+\Delta t)}{m}\Delta t \ , \tag{2}$$

where F_i is the total force acting on the particle, m its mass and Δt is the time step.

A remark about the Euler scheme (2) being *implicit* is in order: Whereas in conventional MD the choice of evaluating the force for the previous or for the new configuration (i.e. at t or $t+\Delta t$ respectively) is merely a matter of stability of the integration scheme (cf. e.g. [19]), the constraints in CD can only be imposed on the *new* configuration. In this sense, the integration scheme of CD is inevitably of implicit type. (Taking into account the configurational change during a time step consistently (*fully implicit* integration [3]) leads to difficulties which are analogous to implicite schemes in MD, when the forces have to be evaluated for the yet unknown new configuration. In one dimension, though, these difficulties do not arise.)

1. One contact

We now turn to the force on the i'th particle, F_i, occurring in Eq. (2): In order to determine it one has to know the contact forces between the grains. In CD they are calculated from the condition that the constraints must not be violated. In one dimension and if one disregards rotations, this is simply the excluded volume contraint. To give a specific example, let us consider two particles with equal masses m subjected to constant external forces (Fig. 1). A contact force R (which is the reaction force due to the constraint) is active only, if interpenetration needs to be prevented. Otherwise, i.e. if the gap g would remain non-negative (no overlap) anyway, it takes on its minimal value, $R=0$ (this is expressed in the CD literature as *Signorini's condition*). Without the repulsion R between the particles the gap at the end of a time step would be given by

$$g' = g_t + (v_{2,t} - v_{1,t})\Delta t - \frac{F_1^{\rm ext} + F_2^{\rm ext}}{m}\Delta t^2 \tag{3}$$

according to (1) and (2). However, the excluded volume constraint requires that $g_{t+\Delta t} = \max\{g', 0\}$. In order that this results from (1) and (2), the contact force

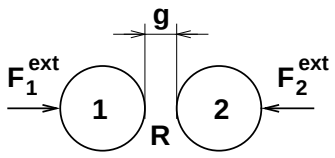

FIG. 1: A pair of particles

$R_{t+\Delta t} = \max\{R', 0\}$ must be taken into account, where

$$R' = -\frac{mg'}{2\Delta t^2} \tag{4}$$

$$= \frac{m}{2\Delta t}\left(\frac{-g_t}{\Delta t} + v_{1,t} - v_{2,t}\right) + \frac{F_1^{\rm ext} + F_2^{\rm ext}}{2} \tag{5}$$

Note that this scheme corresponds to a completely inelastic collision (i.e. with the so-called *restitution coefficient* being zero), which is accomplished in two time steps: At first the gap closes, in the next step also the relative velocity vanishes. (Finite restitution coefficients can also be incorporated into this algorithm [20].)

The above determination of the contact force has a drawback, though: If $g_t < 0$ occurs due to a previous inaccuracy, then the *elimination* of this overlap is accompanied by a surplus of kinetic energy. Therefore, mostly the *quasi-inelastic shock* formula [3]

$$R' = \frac{m}{2\Delta t}\left(\frac{-g_t^{\rm pos}}{\Delta t} + v_{1,t} - v_{2,t}\right) + \frac{F_1^{\rm ext} + F_2^{\rm ext}}{2} \tag{6}$$

is used instead of (5), where $g_t^{\rm pos} = \max\{0, g_t\}$. That means, negative gaps are treated differently from Eq. (5) in such a way that an already existing overlap is not eliminated but only its further growth is inhibited. Hence the inelastic shock law (6) is in a way even "more inelastic" than the original law (5), because it avoids overlap correcting impulses which could destroy stable equilibrium states.

2. Many contacts

We now address the question how to solve the problem of the constraint forces if we consider not only one, but many contacts at the same time. Fig. 2 shows such a system, where R_i denotes the contact force between the

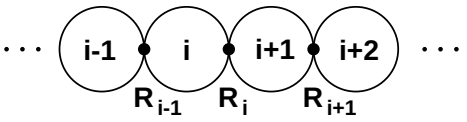

FIG. 2: A multi-contact situation in an 1D array. External forces act only on particles far away from those shown. Each particle is subjected only to the contact forces of the adjacent ones.

particles i and $i+1$, and for the sake of simplicity we have no external force acting on them. Furthermore, we will concentrate on the situation where the neighboring pairs are permanently in contact (i.e. $g^{\rm pos}_{i,t} = 0$) which corresponds to compressed dense packings.

For the ith contact in this setup, R_{i-1} and R_{i+1} play the role of the external forces in Eq. (6), though they are not constant, their values are not known for the next time step:

$$R_{i,t+\Delta t} = \frac{m}{2\Delta t}\left(v_{i,t} - v_{i+1,t}\right) + \frac{R_{i-1,t+\Delta t} + R_{i+1,t+\Delta t}}{2} \ . \tag{7}$$

Obviously the contact force is coupled to the neighboring contacts and through those to further contacts. In a similar way in higher dimensions, large numbers of contacts are coupled within clusters defined by the contact network. Hence, the determination of the proper reaction forces becomes a global problem.

A standard way in CD is to solve this problem iteratively: In one iteration step we calculate the forces according to the constraint conditions pretending that the corresponding neighboring contacts already exhibit the right forces. In that way the process traverses the list of contacts many times until satisfactory convergence is reached (for the question of the convergence cf. the works [21, 22]).

For our one-dimensional chain of particles, this means that Eq. (7) simply gets the meaning of an *assignment* of the right hand side to the left hand side, and one iteration step consists of applying this assignment sequentially once to each contact. The order of this sequence is preferably random (with the pattern changing for every sweep), in order not to create any bias in the information spreading[24]. With each sweep a globally consistent solution is approached, until finally a chosen convergence criterion is fulfilled.

In the next section we shall show, what kind of consequences the local update scheme presented here has on large time and length scales.

III. THE LARGE SCALE DESCRIPTION

In order to analyze the coarse grained behavior of the microscopic equations derived in the last section we can also regard them as the discretized form of a continuum description, making a treatment in terms of partial differential equations (PDE) possible. In order to obtain the corresponding PDEs, we consider the particle index i as space variable x and replace the differences of consecutive quantities by derivatives, the error term for the first and second order derivatives being of first and second order, respectively. E.g. $v_{t+\Delta t} - v_t \to \Delta t\, \partial_t v$ and $R_{i+1} + R_{i-1} - 2R_i \to d^2 \partial_x^2 R$, where d is the particle diameter.

A. The relaxation of the contact forces

While the continuum versions of the updates (1) and (2) can be obtained straight forwardly, the force change (7) lacks a time variable, for during the force-iteration, being just a calculation, no physical time passes. Hence, to be able to describe this force-development as well, let us introduce a fictitious time t^* with time interval Δt^* for one iteration-step. With this, the continuum version of Eq. (7) reads:

$$\partial_{t^*} R = D\partial_x^2 R - \beta \partial_x v \tag{8}$$

$$\text{with} \quad D = q\frac{d^2}{\Delta t^*} \tag{9}$$

$$\beta = q\frac{md}{\Delta t \Delta t^*} \tag{10}$$

$$\text{and} \quad q = \frac{1}{2} \tag{11}$$

This analytic form clearly reveals the nature of the iteration loop: The reaction forces relax towards the solution in a diffusive way. (Note that the $\partial_x v$ term is constant in t^*, it only depends on x.)

The introduction of the constant q reflects a subtlety regarding the sequential character of the update discussed in the appendix. In fact, since the PDE (8) describes the change of the whole field $R(x)$ *at once*, given its actual value at time t^*, it corresponds to a *parallel update* (in the sense that the right hand side of Eq. (7) always employs the values R_i from the *beginning* of the iteration sweep, not the freshly updated ones). In appendix A we shall show, though, that a random sweep update instead of a parallel one only renormalizes the value of q to about 0.8 while leaving the form of the PDE untouched.

B. Sound waves

To connect the velocity update, whose continuum version follows immediately from Eq. (2) as

$$\partial_t v = -\frac{d}{m}\partial_x R \ , \tag{12}$$

to the force update, we must relate the "iteration time" t^* to the physical time t. Although, depending on the convergence criterion, there can be in principle a varying number of iterations during one physical time step Δt, we assume for simplicity this number N_I being fixed. (Actually, in practice this crude "criterion" is sometimes applied.)

Hence, with $\Delta t = N_I \Delta t^*$, we can express everything in terms of the physical time:

$$\partial_t R = D\partial_x^2 R - \beta \partial_x v \ , \tag{13}$$

$$\text{and} \quad D = qN_I\frac{d^2}{\Delta t} \tag{14}$$

$$\beta = qN_I\frac{md}{\Delta t^2} \tag{15}$$

With the equations (12) and (13) we obtained two coupled PDEs. We can combine them to arrive at a wave equation with an additional damping term:

$$\partial_t^2 R = c^2 \partial_x^2 R + \partial_t \left(D \partial_x^2 R \right) \tag{16}$$

The sound velocity appearing is of finite value:

$$c = \sqrt{q N_I} \frac{d}{\Delta t} \tag{17}$$

This equation indicates that the CD simulation of the particle chain, as presented in the previous section, can lead to sound propagation like in an elastic medium, which however contradicts the conception of perfect rigidity. The constraint conditions applied at the contacts should in principle prohibit overlaps, i.e. prohibit elastic deformation of the grains. It can be seen that this deviation from the perfect rigidity enters at the force relaxation: A finite number of iterations means a finite range for the information spreading and thus yields systematic errors in the calculated reaction forces. As a consequence, the finite N_I involves soft particles and a finite sound velocity $c \sim \sqrt{N_I}$. Note that in the limit of an infinite N_I, the exact value of the forces is reached, which corresponds to the case $c \to \infty$, as it should be for rigid particles.

1. Dispersion

Performing a Fourier transformation on Eq. (16), one obtains the properties of the different wave modes. The oscillation frequency ω of the wave number k is

$$\omega(k) = k \sqrt{c^2 - \frac{D^2 k^2}{4}} \ . \tag{18}$$

That means, $\omega(k)$ becomes zero at a critical wave number

$$k_c = \frac{2c}{D} \sim \frac{1}{\sqrt{N_I}} \ , \tag{19}$$

and waves with k larger than k_c (short wave lengths) are over-damped. The damping time $\tau(k)$ for the oscillating modes is given by:

$$\tau(k) = \frac{2}{D k^2} \tag{20}$$

We derived the dispersion relation (18) in the continuum limit which is a good approximation for small wave numbers, but not near to the border of the Brillouin zone ($k_{\mathrm{Br}} = 2\pi/d$), where the effect of the spatial discreteness can be strong. However, increasing the number of the iterations sufficiently, k_c becomes small compared to k_{Br}. Actually, for $N_I \geq 10$ the formula (18) works well not only for small wave numbers but for all the oscillating modes, as could be verified numerically.

2. Numerical confirmation

In order to confirm the results of this section, we performed the following numerical experiment: The starting configuration of the simulation consists of an array of 50 discs and an immobile wall, the geometry can be seen in Fig. 3. Initially the gap between the wall and the leftmost particle is one disc diameter (d), the gap between the particles is zero and the array has zero velocity. Starting from $t = 0$ a constant external force (F^{ext}) is acting on the rightmost particle which accelerates the array towards the wall (only horizontal motion takes place). As simulation parameters we chose $N_I = 40$ and $F^{\mathrm{ext}} = 0.05\, dm\Delta t^{-2}$.

The collision with the wall induces a relative motion of the grains and generates sound waves in the array. After a transient period the grains remain permanently in contact (the whole array is pressed against the wall by F^{ext}). Since the different wave modes have different relaxation time, after a while only the largest wave length mode survives. This wave length is four times the system size because the wall represents a fixed boundary while the right side is free. Since the wave length is given, the oscillating frequency and the damping time can be calculated from Eq. (18) and Eq. (20), respectively. For comparison with the simulation we measured the motion of the rightmost particle. The expected motion is a damped oscillation

$$x(t) = x_0 + A \exp(-t/\tau) \sin(\omega t + \phi) \ , \tag{21}$$

where the offset x_0, the amplitude A and the phase shift ϕ have to be fitted (in contrast to ω and τ) for a comparison. In Fig. 4 the measured data (dots) and the fitted curve can be seen. It shows that the simulation is in good agreement with our continuum description.

C. Global Elasticity

It is instructive to compare our test-system to its simplest MD counterpart where the contact forces depend linearly on the local kinematic variables, i.e. the so called *linear spring/dashpot model*

$$R_i = -\kappa (x_{i+1} - x_i - d) - \gamma (v_{i+1} - v_i) \tag{22}$$

with the spring stiffness κ and the damping coefficient γ. Employing again the updates (1) and (2) for the positions and velocities, respectively, the continuum limit yields

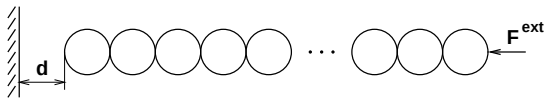

FIG. 3: The initial configuration of the numerical experiment for testing the properties of the sound waves.

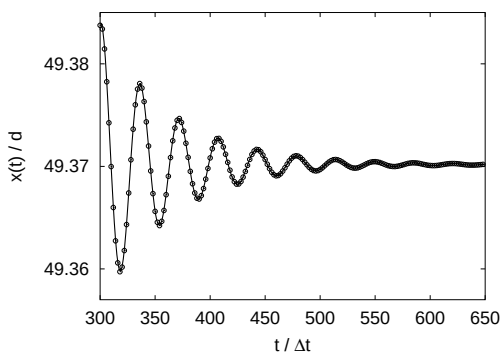

FIG. 4: Damped oscillation in a Contact Dynamics simulation. The dots indicate the measured data: the position of the rightmost particle versus time (for details see the text). The line is an exponentially damped sinus function, where the frequency and the damping time is provided by our continuum model.

the same type of PDE as Eq. (13) with its coefficients being inherited from Eq. (22):

$$\partial_t R = \frac{\gamma d^2}{m}\partial_x^2 R - \kappa d \partial_x v \tag{23}$$

This allows us to relate the physical MD model parameters to the "technical" CD parameters:

$$\kappa = qm\frac{N_I}{\Delta t^2} \tag{24}$$

and

$$\gamma = qm\frac{N_I}{\Delta t} \tag{25}$$

This equivalence shows that on large scales the CD chain should behave identical to its MD counterpart, e.g. it will exhibit a global shrinkage proportional to an external compressive load. Note that a real congruence can be expected only for the collective behavior but not on the level of the contacts. In the CD method, as explained above, the contact forces are not related to the overlaps, which must merely be regarded as due to the incompleteness of the force-calculation and in fact are stochastic quantities because of our random update procedure. Only on scales larger than the grain size, where the fluctuations of these local "deformations" are averaged out, the behavior can be smooth like in an elastic medium, as is shown in Fig. 4.

In sections III B and III C, our calculation was based on the assumption of a constant number of iterations for every time step, and due to this premise the analytical treatment became simple and directly comparable to the corresponding simulation. We should keep in mind, though, that the application of a convergence criterion involves a fluctuating N_I (i.e. it can vary from time step to time step), and therefore steps with a different "stiffness" are mixed during the integration of motion. Consequently, the behavior of the CD method is more complex in detail, but qualitatively the results for the constant N_I remain relevant also here. (E.g. the mechanism resulting in soft particles or the way how shock-waves can arise with finite velocity.)

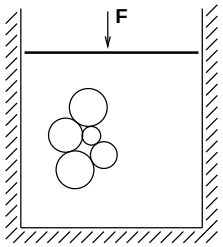

FIG. 5: The setup of a numerical experiment in two dimensions. A dense packing of 1000 discs is prepared in a container via compressing the system by means of the mobile upper wall.

IV. 2D SIMULATION

After the analysis of the regular 1D system, the important question arises whether the behavior is similar in higher dimensions and less regular systems. Hence, we performed CD simulations with two-dimensional random packings of discs and observed the same "elastic" waves (even transversal modes were found).

The simulation presented here consists of 1000 discs with radii distributed uniformly between $r_{\min}$ and $r_{\max} = 2r_{\min}$, the mass of each disc being proportional to its area. Fig. 5 shows the geometry: The base and the two side-walls are fixed while the upper piston is mobile. Starting from a loose state, we compressed the system and waited until the packing reached an equilibrium state (the compression force F applied on the piston was kept constant). The simulation was carried out without gravity and with a Coulomb friction coefficient of 0.05 for all the disc-disc and disc-wall contacts (cf. [23]).

After the packing was relaxed completely, we generated sound waves by increasing the compression force abruptly to $F + \Delta F$. After a transient period only one standing wave mode survives (both the wavenumber vector and the collective motion are vertical), where the piston, representing a free boundary, oscillates with a relatively large amplitude. We measured the vertical position of the piston versus time and found that the data can be fitted by an exponentially damped sine function (Fig. 6). Here, in contrast to the 1D case, also ω and τ are fit parameters, since, due to the different geometry, the values (18) and (20) cannot be adopted, but, because of the random structure of the system, a more complex treatment is required for a quantitative description. However, we checked the most important relation, namely that the

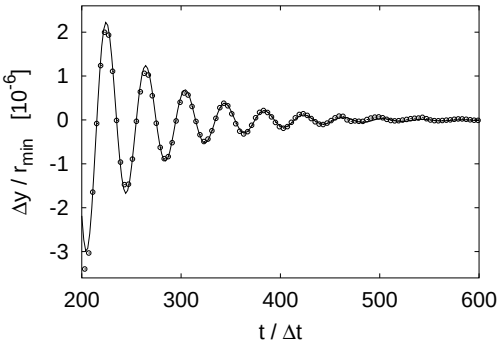

FIG. 6: Oscillations in a 2D simulation are similar to the 1D case. Here the sound waves are generated in a random dense packing of discs. The dots are the measured position of the upper wall versus time (see Fig. 5), while the curve is a fitted exponentially damped sinus function.

scaling properties of ω and τ remain valid also for the 2D random system, that is $\omega \sim \sqrt{N_I}$ and $\tau \sim N_I^{-1}$, which means that the artificial visco-elasticity of the particles depends on the number of the iterations in the same manner as we showed for the 1D chain.

V. DISCUSSION

The artificial elasticity found in CD simulations was analyzed. We showed that the systematic errors of the force calculation can lead to a collective elastic behavior, even though single contacts are assumed perfectly rigid and perfectly inelastic. For the 1D chain of particles, we could, starting from the "microscopic" laws, reproduce the numerical results analytically, including the dependence of the effective stiffness and viscous dissipation of the contacts on the computational parameters $(N_I, \Delta t)$.

Besides elucidating the origin of the elastic behaviour, the coarse grained description reveals important characteristics of the CD method, which were less obvious on the discrete level. We saw that using the iterative solver, the proper contact forces are approached in a diffusion like manner which is a crucial information concerning the computational time. The conception of perfectly rigid particles requires that the calculated forces are consistent even for contacts far from each other. Therefore, the "diffusion length" $\sqrt{D\Delta t}$ must be larger than the linear system size L, which defines a lower boundary for the number of iterations of the order $(L/d)^2$. The same condition is obtained if we want to avoid other consequences of the effective softness. E.g. if we want the sound to travel a larger distance than L during one time step (i.e. $c > L/\Delta t$) or if we would like all possible wave modes to be overdamped (i.e. with $\lambda_c > L$ no oscillations). All these cases are equivalent, one is forced to apply a relatively large number of iterations: $N_I \sim L^2$. Going further on this line, the scaling with respect to the number of particles n can be determined: One step of the iteration consists of as many force-updates as there are contacts, which is proportional to n. Therefore, the computational effort of one time step scales with the particle number like nN_I, which is $\sim n^2$ in 2D or $\sim n^{5/3}$ in 3D. Therefore a large CD-simulation is computationally more costly than MD, where the computational effort scales like n. This is the price for simulating rigid particles without getting elasticity artifacts, which cannot be done with MD.

To avoid this super-linear scaling when dealing with large systems, we can also accept the finite stiffness by keeping N_I constant independently of n. Then, besides gaining a running time of order n, of course, elastic deformations and sound waves can arise with an increasing number of the particles, and consequently they have to be monitored. We want to mention the idea, though, that in certain situations advantage can be taken of the artifact. E.g. when being applied deliberately, Coulombian friction can be combined with global elasticity easily; this way considerable computational time could be saved and even better performance than MD could be achieved.

VI. ACKNOWLEDGMENTS

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APPENDIX A: q FOR THE RANDOM SWEEP UPDATE

To find the proper value of the constant q appearing in sec. III, two more things have to be taken into account: Firstly the sequential type of the force update, and secondly that the order is random. The latter results in a stochastic force relaxation, i.e. the change of R_i in one iteration sweep is a stochastic variable. In order to obtain a similar equation as (8) we shall determine the average value $\langle \Delta R_i \rangle$, where the average for site i is meant to be taken over all possible update sequences.

Before going any further, let us introduce a few notations:

- For N being the total number of contacts, the mapping $u : \{1, \ldots, N\} \to \{1, \ldots, N\}$ denotes the order of the update sequence; i.e. if the contact labeled i is updated before j, then $u_i < u_j$.
- Throughout this appendix, the notation R_i means the value from the beginning of the iteration sweep, the value at the end is $R_i + \Delta R_i$.
- We define δR_i as the change according to a parallel

update (cf. Eq. (7)), i.e.

$$\delta R_i = \frac{m}{2\Delta t}\,(v_i - v_{i+1}) + \frac{R_{i-1} + R_{i+1} - 2R_i}{2}\ , \qquad \text{(A1)}$$

as opposed to the *total* change ΔR_i.

Given that, it can easily be seen how ΔR_i depends on the update order. If e.g. the site i is updated earlier than its neighbors (i.e. $u_{i-1} > u_i < u_{i+1}$), then $\Delta R_i = \delta R_i$, but in the case $u_{i-1} > u_i > u_{i+1} < u_{i+2}$, we get $\Delta R_i = \delta R_i + \delta R_{i+1}/2$ (because for contact i, Eq. (7) employs the already updated force at contact $i+1$). Similarly (but with less probability) even very far contacts can contribute to ΔR_i, which can be summarized in the following way:

$$\langle \Delta R_i \rangle = \delta R_i + \sum_r \frac{1}{2^r}\,(p_r \delta R_{i-r} + p_r \delta R_{i+r}) \qquad \text{(A2)}$$

Here, p_r is the probability that ΔR_i contains information from the update of a contact at distance r, that is $p_r = P(u_i > u_{i+1} > \ldots > u_{i+r})$. (This definition holds true for contributions from contacts with labels higher than i, but due to left-right symmetry the same value is inevitably obtained for the corresponding lower ones.)

The value of p_r can be obtained from the following combinatorial consideration: Given an index i and a distance r, we can classify the set of all update orders into groups such that the sequences in one group differ only in the permutations of the elements u_j, $i \le j \le i+r$. Such a group contains $(r+1)!$ sequences, but only one of them satisfies the condition $u_i > u_{i+1} > u_{i+2} > \ldots > u_{i+r}$. Since all update sequences are equally probable, the value of p_r is equal to $1/(r+1)!$.

The factor $(2^r(r+1)!)^{-1}$, relating the contacts i and $i+r$, decays faster than exponentially; already for $r = 8$, it drops below 10^{-6}. Therefore, the sum in Eq. (A2) reaches only the immediate vicinity of contact i, such that, for our large wavelength considerations, the approximation $\delta R_{i+r} \approx \delta R_i$ can be applied. This allows us to calculate the average change of the contact force:

$$\langle \Delta R_i \rangle = \delta R_i \left(1 + 2\sum_{r=1}^{\infty} \frac{1}{2^r (r+1)!}\right) = \delta R_i \left(4\sqrt{e} - 5\right)\ . \qquad \text{(A3)}$$

Thus, it is shown that the random sweep results in a larger change of R_i than the parallel update. Eq. (A3) provides also the sought value of the parameter q as

$$q = \frac{4\sqrt{e} - 5}{2} \approx 0.797\ , \qquad \text{(A4)}$$

which completes the continuum description given in sec. III.

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[24] We call this procedure "random sweep" as opposed to "random sequential" where the choice of the contact to be updated would be random *and independent* of the previous choices.