# Extent of force indeterminacy in packings of frictional rigid disks

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Static packings of frictional rigid particles are investigated by means of discrete element simulations. We explore the ensemble of allowed force realizations in the space of contact forces for a given packing structure. We estimate the extent of force indeterminacy with different methods. The indeterminacy exhibits a nonmonotonic dependence on the interparticle friction coefficient. We verify directly that larger force indeterminacy is accompanied by a more robust behavior against local perturbations. We also investigate the local indeterminacy of individual contact forces. The probability distribution of local indeterminacy changes its shape depending on friction. We find that local indeterminacy tends to be larger on force chains for intermediate friction. This correlation disappears in the large friction limit.

In packings of relatively rigid particles, elastic deformations of the grains are typically several orders of magnitude smaller than the grain size. Since this separation of length scales occurs it is a natural idea to investigate the limit case of infinite stiffness of the grains.

It is known that jammed packings of perfectly rigid particles with finite friction coefficient are "*hyperstatic*" [1,2]. The number of equations of mechanical balance is smaller than the number of unknowns (components of the interparticle forces). This makes the problem undetermined in the sense that there are many solutions that satisfy the equilibrium equations. Even taking constraint conditions, such as Coulomb's limit of friction and unilaterality of the contacts, into account does not help to eliminate the indeterminacy of the contact forces. Thus for a given packing geometry the solutions define an ensemble of admissible force networks $S$ [3,4]. $S$ is a convex set [5] in the force space $\mathcal{F}$, spanned by the components of contact forces, and its boundaries are delimited by constraint conditions.

The ensemble has received considerable attention since many macroscopic properties of granular packings can be derived from ensemble averaging over all allowed force states supposing a uniform measure on $S$ [6–11]. Furthermore, with this technique one can disentangle the effect of forces and texture of the packing. Mathematically, the problem of finding the solutions of a set of undetermined equations and constraints is of rather broad interest, e.g., in metabolic networks [12,13].

The extent of force indeterminacy in two-dimensional (2D) random packings of perfectly rigid disks was investigated theoretically and numerically in [14]. The indeterminacy of each component of the contact forces was obtained, suggesting that highly undetermined contacts are located on main force chains. Force indeterminacy in such packings was also measured in [5] where it turned out that the indeterminacy depends nonmonotonically on the interparticle friction coefficient due to the competition between two coexisting effects: the opening of the Coulomb cone angle and the lowering of connectivity. In Ref. [15] similar nonmonotonic fric-

tion dependence is obtained for mechanical response of the granular packings to local perturbations.

In this Brief Report we examine whether the nonmonotonic friction dependence of force indeterminacy remains valid also when other methods are used to quantify the "size" of the solution set $S$. We measure numerically the extent of force indeterminacy and the mechanical response to local perturbations in the same packings and examine the relation between them. The local force indeterminacy is also studied in this work. First, we investigate its probability distribution, then, we compare its spatial pattern to that of the force chains in the packing.

*Sampling procedure*. The systems we investigate are 2D random packings of 400 perfectly rigid disks. Periodic boundary conditions are applied in both directions, disk radii are uniformly distributed between 0.5 and 1, gravity is set to zero, and the unit of the length is set to the maximum grain radius. Our numerical simulations consist of two steps which are performed with the help of contact dynamics algorithm [16–18]. First we construct static configurations of particles. The initial dilute systems are compressed by imposing a homogeneous confining pressure $P_0$ to get the final static packings. The full description of our method of constructing the homogeneous packings can be found in [19].

Then we explore the force ensemble: we collect force networks that provide static solutions for the given contact geometry and boundary conditions. We use a random-walk method in the force space [5,20] starting with the original force network. We perturb the original force state and jump to a new force state in the force space $\mathcal{F}$. The technique is to add random values that are chosen uniformly from the interval $[-\langle F_n \rangle, \langle F_n \rangle]$ to all components of the contact forces. $\langle F_n \rangle$ is the mean normal force calculated over the current values of contact forces. The perturbed force network is given as the input for the Gauss-Seidel-type iterative solver of the contact dynamics method which lets the forces relax into a new consistent state. The jump is accepted if the new state is an equilibrium state, otherwise it is rejected. The perturbation and relaxation are repeated many times, always starting from the last equilibrium force network. In this way we collect

1000 admissible force networks for a given static packing. In order to study systematically the influence of the interparticle friction coefficient on the extent of the force indeterminacy, the constructing and sampling procedures are repeated for various values of the friction coefficient.

*Numerical results.* We first quantify the extent of the force indeterminacy $\eta$ for a given packing geometry based on the sampled force networks. We compare here three different methods, since there is no *a priori* preferred way to measure $\eta$. Let us denote the center of the samples in the force space $\mathcal{F}$ by $\{\vec{G}_c\}$ which is a force network with contact force vectors $\vec{G}_c$ given by

$$\vec{G}_c = \langle \vec{F}_c \rangle_{\text{states}}, \quad c = 1, \ldots, N_c, \qquad (1)$$

where the average $\langle \cdots \rangle_{\text{states}}$ is taken over all realizations of the force states and $N_c$ is the number of contacts. One possibility to quantify the force indeterminacy is to measure the force fluctuations $\delta F_c$ around the mean force vector $\vec{G}_c$ at each contact $c$ [5],

$$\delta F_c = \langle (\vec{F}_c - \vec{G}_c)^2 \rangle_{\text{states}}^{1/2}. \qquad (2)$$

The force indeterminacy $\eta_1$ of the whole packing is given by the relative fluctuation

$$\eta_1 = \frac{\langle \delta F_c \rangle_{\text{cont.}}}{\langle |\vec{G}_c| \rangle_{\text{cont.}}}, \qquad (3)$$

where $\langle \cdots \rangle_{\text{cont.}}$ denotes the average over all contacts.

The extent of the indeterminacy could be also estimated by the Euclidean distance between randomly chosen pairs of force states [7,14]. The probability distribution of the distances becomes sharply peaked if $\mathcal{S}$ is a high dimensional object. The global indeterminacy according to this method is defined via

$$\eta_2 = \left( \frac{\langle (\{\vec{F}_c\} - \{\vec{F}_c\}')^2 \rangle_{\text{pairs}}}{\{\vec{G}_c\}^2} \right)^{1/2}, \qquad (4)$$

where $\{\vec{F}_c\}$ and $\{\vec{F}_c\}'$ are two different force states and $\langle \cdots \rangle_{\text{pairs}}$ means the average over all pairs of force states. The square of a force state $\{\vec{F}_c\}^2$ is given by $\Sigma_c \vec{F}_c^2$.

As an alternative method [14], the extremal points of $\mathcal{S}$ along each axis of the force space $\mathcal{F}$ provide the following measure of the indeterminacy:

$$\eta_3 = \frac{\langle F^{\text{max}} - F^{\text{min}} \rangle_{\text{comp.}}}{\langle \frac{F^{\text{max}} + F^{\text{min}}}{2} \rangle_{\text{comp.}}}. \qquad (5)$$

Here, $F^{\text{max}}$ and $F^{\text{min}}$ are the maximum and minimum values of a contact force component (either normal or tangential). The average $\langle \cdots \rangle_{\text{comp.}}$ is taken over all $2 \times N_c$ components of contact forces. We note that these are mathematically three different measures that give different results depending on the circumstances, e.g., the first two methods [Eqs. (3) and (4)] depend on the probability measure that is realized by the sampling. This is not the case for $\eta_3$ which has a pure geometrical definition (provided the sampling explores the solution set). The question whether the sampling is uniform or
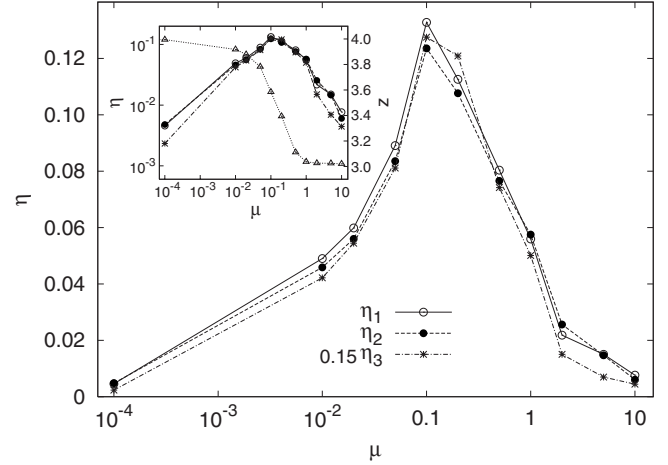


FIG. 1. Force indeterminacy $\eta$, quantified with different methods that are explained in the text, in terms of the friction coefficient $\mu$. Inset displays the same plot in log-log scale. The average coordination number $z$ as a function of $\mu$ is also shown (triangles).

not has no effect on the value of $\eta_3$. Our numerical tests (not shown) reveal that the values of $\eta_i$ are well reproducible and do not strongly depend on the total number of grains.

In Fig. 1 we compare the values of $\eta$ obtained by the three methods which, up to a constant factor, provide basically the same behavior in the whole range of friction ($\eta_1 \approx \eta_2 \approx 0.15 \eta_3$). The nonmonotonic friction dependence, reported in [5], is reproduced here independently of the quantifying method. We note that the average coordination number $z$ of the packing depends strongly on the friction coefficient (inset of Fig. 1); consequently, the dimension of $\mathcal{S}$ is varied with $\mu$.

Next we investigate the effect of $\eta$ on the mechanical response of granular packings. In Refs. [15,21] local perturbations were used to break the equilibrium structure of the homogeneous packings and induce motion of the grains. It turned out that the displacements of the particles due to local perturbations decay as a power law of the distance from the perturbation point. The numerical experiment was repeated for several packings constructed with different $\mu$ revealing that the decay exponent $\alpha$ and the critical force $F_{\text{crit}}$, i.e., the force needed to break the mechanical equilibrium, exhibit a nonmonotonic dependence on the friction with extrema at $\mu = 0.1$ similarly to the behavior of $\eta$. This similarity suggests the picture that a packing with larger force indeterminacy becomes more stable against perturbations. Here, we test directly whether such a relation exists. Together with the force indeterminacy we determine also the response quantities $F_{\text{crit}}$ and $\alpha$ for the same packing configurations. Since the different methods we used to quantify $\eta$ are basically equivalent, we plot the response quantities in terms of $\eta_2$ in Fig. 2. The same series of packings are plotted here as in Fig. 1. The lines are connecting the data points in the order of increasing friction. Both $F_{\text{crit}}$ and $\alpha$ are strongly related to the extent of force indeterminacy, although they are not a unique function of $\eta$. Still, very different packings (with different density, connectivity, and frictional properties) exhibit similar response properties if their $\eta$ values are close to each other.

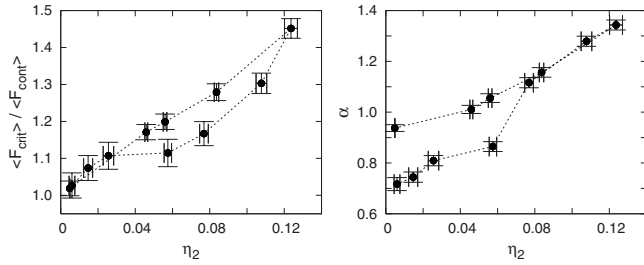The quantity $\eta$, as a global parameter, provides a good

FIG. 2. The average critical force $\langle F_{\mathrm{crit}} \rangle$ scaled by the average normal contact force $\langle F_{\mathrm{cont}} \rangle$ (left) and the decay exponent $\alpha$ (right) as functions of the force indeterminacy $\eta_2$ for packings that are constructed with different friction coefficients.

estimate of the size of $\mathcal{S}$. In order to learn more details on the structure of the solution set we study the indeterminacy also locally at the level of individual contacts. In Fig. 3 we show the values of normal contact forces $F_n$ for every contact obtained by the sampled realizations of force states. We note that the data points are highly correlated, the contact forces cannot vary independently of each other, but here we investigate only how the solution set $\mathcal{S}$ is elongated along each "$F_n$" axis. At each contact the possible values of $F_n$ form an interval $\Delta F_n$ because these values are the projection of the convex solution set $\mathcal{S}$ onto the $F_n$ axis. The length of the interval $\Delta F_n$ serves as a measure of the local indeterminacy at a given contact and can be estimated with the help of extrema of $F_n$ that were provided by the sampling procedure.

Figure 3 reveals that the fluctuations of the lengths $\Delta F_n$ grow nonmonotonically with friction similarly to the behavior of $\eta$. Moreover, it can be also seen from Fig. 3 that the length of $\Delta F_n$ varies widely from one contact to another, even for a given friction coefficient. We now aim to investigate how frequent the different magnitudes of the local indeterminacies are. In Fig. 4 we show the probability distribution of $\Delta F_n$ for different friction coefficients. $P(\Delta F_n)$ is a monotonically decreasing function for small and large friction limits, but becomes broader and displays a peak for intermediate friction coefficients. This shows that the value of friction coefficient has even a geometrical signature in the solution set. Intermediate values of $\mu$ provide a more compact shape of $\mathcal{S}$ in the sense that the frequency of extreme



FIG. 4. The probability distribution of the interval of possible normal contact forces for several friction coefficients $\mu$. Insets display semilogarithmic probability distributions for three different $\mu$.

dimensions are suppressed compared to the case of small and large $\mu$ where we find more pronounced extremes and therefore a more elongated shape of the solution set. The tail of $P(\Delta F_n)$ follows exponential decay for small frictions, while it decays faster (slower) than exponential for intermediate (large) frictions (see the insets of Fig. 4).

Finally, we investigate the spatial distribution of the indeterminacy throughout the system. The aim is to find whether contacts that are located in a main force chain carry also larger force indeterminacies. In Fig. 5 (full circles) we plot the location of contacts that have larger $F_n$ than twice of the average normal contact force $\langle F_n \rangle$ (according to the original force network in the packing). We also plot the contacts with large force indeterminacy $\Delta F_n$ (open circles) above the average $\langle \Delta F_n \rangle$. This way approximately the same number of open and full circles is plotted. It can be seen that contacts in force chains tend to have larger force indeterminacy in case of intermediate friction [Fig. 5(a)], but for $\mu=10$ the two patterns become seemingly different [Fig. 5(b)]. Indeed, if we determine the correlation between $\Delta F_n$ and $F_n$ over all
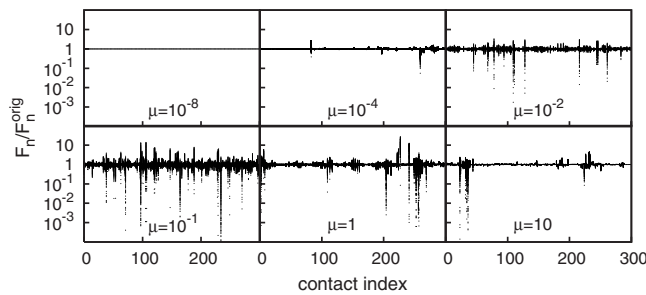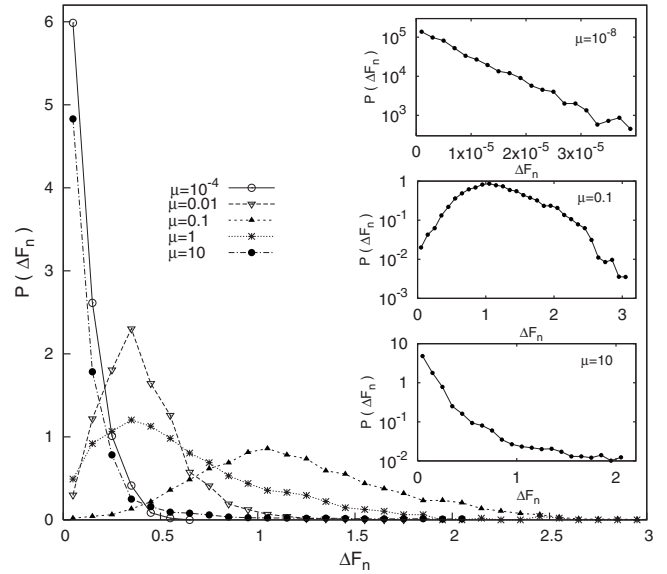


FIG. 3. The possible values of the normal contact forces $F_n$ (denoted by dots and mostly merged to intervals), scaled by the normal component of the original contact force $F_n^{\mathrm{orig}}$, are shown for each contact of the packing. Each plot corresponds to the packing with the given friction coefficient. The fluctuations are small for small and large $\mu$ and become larger at intermediate $\mu$.
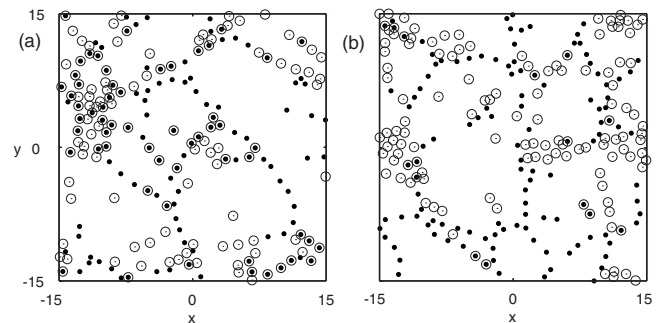


FIG. 5. The position of the contacts with original normal force $F_n$ larger than $2\langle F_n \rangle$ (●) and the position of the contacts with force indeterminacy $\Delta F_n$ larger than $\langle \Delta F_n \rangle$ (⊙) in the packing with (a) $\mu=0.1$ and (b) $\mu=10$.
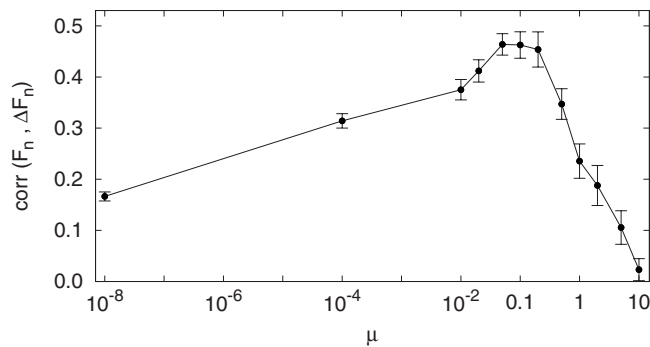
FIG. 6. The correlation between the normal component of the original contact force $F_n$ and the force indeterminacy $\Delta F_n$ in terms of the friction coefficient $\mu$. Error bars display the maximum possible change in the correlation values when an arbitrary force network is chosen.

contacts of the original force network and plot it against the friction coefficient (Fig. 6), it reveals that the correlation vanishes for large frictions. Interestingly, the correlation exhibits again a nonmonotonic dependence on friction, where significant correlations are present for the intermediate friction regime and weaker correlations outside. This may have a statistical origin. When the indeterminacy is large (intermediate friction) it is plausible to expect that the correlation between $\Delta F_n$ and $F_n$ is enhanced, since finding a large force is more probable in a large interval of positive normal forces.

We note that local force indeterminacies can be seen everywhere in our packings. This is in contrast to what has been reported in Ref. [5], where in case of large friction undetermined contacts formed localized clusters while contact forces in the rest of the packing became uniquely determined. Such localization is absent in the present study. We think that this difference originates from the different boundary conditions. In [5] the forces were kept fixed at the boundary which furthers the formation of a fully determined region. This is not the case here, where we prescribe only the global pressure.

*Conclusion*. In this Brief Report we presented the numerical results of the measurement of force indeterminacy in packings of frictional hard disks. We quantified the global force indeterminacy $\eta$ of the packing with different methods and systematically studied the effect of interparticle friction coefficient. $\eta$ depends nonmonotonically on friction. We showed that the extent of force indeterminacy has an important influence on the mechanical response properties of the material.

The indeterminacy was also studied locally at the level of individual contacts. As a consequence, we observed a nonmonotonic influence of the friction on the shape of the solution set. Small and large values of the friction coefficient enhance the relative weight of extreme dimensions and lead to a more elongated solution set in the force space. We found significant correlation between the spatial pattern of the force indeterminacy and force chains for intermediate friction, however, the correlation disappeared in the large friction limit.

[1] J. N. Roux, Phys. Rev. E **61**, 6802 (2000).

[2] L. E. Silbert, D. Ertas, G. S. Grest, T. C. Halsey, and D. Levine, Phys. Rev. E **65**, 031304 (2002).

[3] J. H. Snoeijer, T. J. H. Vlugt, M. van Hecke, and W. van Saarloos, Phys. Rev. Lett. **92**, 054302 (2004).

[4] J. H. Snoeijer, T. J. H. Vlugt, W. G. Ellenbroek, M. van Hecke, and J. M. J. van Leeuwen, Phys. Rev. E **70**, 061306 (2004).

[5] T. Unger, J. Kertész, and D. E. Wolf, Phys. Rev. Lett. **94**, 178001 (2005).

[6] S. Ostojic and D. Panja, Europhys. Lett. **71**, 70 (2005).

[7] J. H. Snoeijer, W. G. Ellenbroek, T. J. H. Vlugt, and M. van Hecke, Phys. Rev. Lett. **96**, 098001 (2006).

[8] S. Ostojic and D. Panja, Phys. Rev. Lett. **97**, 208001 (2006).

[9] A. R. T. van Eerd, W. G. Ellenbroek, M. van Hecke, J. H. Snoeijer, and T. J. H. Vlugt, Phys. Rev. E **75**, 060302(R) (2007).

[10] S. Ostojic, T. J. H. Vlugt, and B. Nienhuis, Phys. Rev. E **75**, 030301(R) (2007).

[11] W. G. Ellenbroek and J. H. Snoeijer, J. Stat. Mech.: Theory Exp. (2007) P01023.

[12] D. Segre, D. Vitkup, and G. M. Church, Proc. Natl. Acad. Sci. U.S.A. **99**, 15112 (2002).

[13] E. Almaas, B. Kovács, T. Vicsek, Z. N. Oltvai, and A. L. Barabási, Nature (London) **427**, 839 (2004).

[14] S. McNamara and H. Herrmann, Phys. Rev. E **70**, 061303 (2004).

[15] M. Reza Shaebani, T. Unger, and J. Kertesz, Phys. Rev. E **76**, 030301(R) (2007).

[16] J. J. Moreau, Eur. J. Mech. A/Solids **13**, 93 (1994).

[17] M. Jean, Comput. Methods Appl. Mech. Eng. **177**, 235 (1999).

[18] L. Brendel, T. Unger, and D. E. Wolf, *The Physics of Granular Media* (Wiley, New York, 2004), pp. 325–343.

[19] M. R. Shaebani, T. Unger, and J. Kertész, e-print arXiv:0803.3566, Int. J. Mod. Phys. C (to be published).

[20] T. Unger and J. Kertész, Int. J. Mod. Phys. B **17**, 5623 (2003).

[21] M. R. Shaebani, T. Unger, and J. Kertész, Phys. Rev. E **78**, 011308 (2008).